Landau theory approach to Ising Models.

Consider ferromagnetic Ising model on the square lattice: $H = -J\sum_{\langle ij\rangle} s_i s_j$. Fourier transforming

$$s_i = \sum_k s_k e^{ik\cdot r_i}. \quad H = -2J\sum_k |s(k)|^2 [\cos(k_x) + \cos(k_y)]$$

This looks like a guassian theory, but it is not. We need to impose the constraint $s_i^2 = 1$ which couples different k-modes: $\sum_{kk'} s_k s_{k'} e^{i(k+k')\cdot r_i} = 1$.

Let's ignore this constraint at first.

H has a single minimum at $\vec{k} = \vec{0}$. This indicates that the ground state is translationally invariant i.e. a ferromagnet:

↑ ↑ ↑
↑ ↑ ↑
↑ ↑ ↑

One can write,

$$S(r) = e^{i\vec{0}\cdot\vec{r}} \phi(\vec{r})$$

where $\phi(\vec{r})$ is a slowly varying real mode. Thus the low energy theory is $\int d^dx \left[(\nabla\phi)^2 + \phi^2 + \phi^4\right]$. This implies that the transition from ferromagnet to paramagnet will be in the Ising universality class.

Next consider an antiferromagnetic Ising model:

$$H = +J \sum_{\langle ij \rangle} S_i S_j \quad \text{on square lattice with } J>0.$$

Again, Fourier transforming $H = 2J \sum_k |s(k)|^2 [\cos(k_x) + \cos(k_y)]$.

Now the minima lies at $(k_x, k_y) = (\pi, \pi)$, indicating that the ground state looks like

$$\begin{matrix} \uparrow & \downarrow & \uparrow \\ \downarrow & \uparrow & \downarrow \\ \uparrow & \downarrow & \uparrow \end{matrix}$$

One can now write

$$S(r) = e^{i \vec{K} \cdot \vec{r}} \phi(\vec{r}) \quad \text{where}$$

$\vec{K} = (\pi, \pi)$ and $\phi(\vec{r})$ is a slowly varying real field. Therefore, the low energy theory is: $\int (\nabla \phi)^2 + \phi^2 + \phi^4$, same as ferromagnet. Again, this implies that the transition from AFM to paramagnet will be in the Ising universality.

Fully Frustrated Ising Model on Square lattice

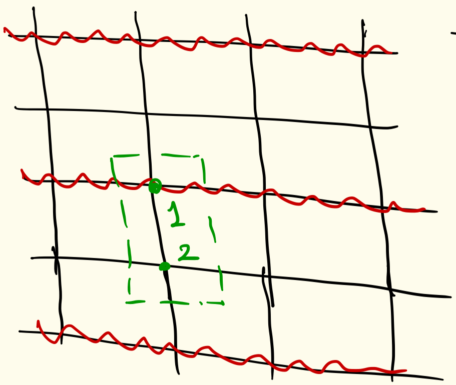
As discussed before, odd Ising guage theory (or, the model $-\sum\prod_{\Box} Z + \sum\prod_{+} X - h\sum X$) is dual to fully frustrated transverse field Ising model:

$$H = \sum_{\langle ij\rangle} J_{ij}\, S^z_i S^z_j - h\sum S^x_i \quad \text{where}$$

J_{ij} satisfy $\prod_{\Box} J_{ij} = -1$. This is a frustrated model because when $h=0$, there are infinitely many ground states (on the other hand, were $\prod_{\Box} J_{ij} = 1$, system orders ferromagnetically and there are just two ground states). One can think of J_{ij} as a static $\mathbb{Z}_2$ guage field and S_i as $\mathbb{Z}_2$ charges. The condition $\prod_{\Box} J_{ij} = -1$ means that there is a $\mathbb{Z}_2$ flux through every plaquette. To make progress, let's chose a gauge so that the condition $\prod_{\Box} J_{ij} = -1$ is satisfied..

The crucial point to note is that although the gauge choice may break lattice symmetries, the actual system still has full symmetries of the square lattice (translation by $\hat{x}, \hat{y}$ and rotation by $\pi/2$)

Lets chose the following gauge:



The red links correspond to $J_{ij} = +1$ and black ones $J_{ij} = -1$

We will use a Landau theory approach, similar to the one discussed above. Lets first set $h = 0$. Chosing the unit cell as indicated above, the J term is:

$$-\sum_{r} \left[S_1(\vec{r})\, S_2(\vec{r}) - S_1(\vec{r})\, S_1(\vec{r}+\hat{x}) + S_2(\vec{r})\, S_2(\vec{r}+\hat{x}) + S_1(\vec{r})\, S_2(\vec{r}+\hat{y}) \right]$$

$$= -\sum_{k} S_1(k)\, S_2(-k)\, [1 + e^{ik_y}] - S_1(k)\, S_1(-k) e^{ik_x} + S_2(k) S_2(-k)\, e^{ik_x}$$

$$= -\sum_k \Big[S_1(k) S_2(-k) \frac{[1+e^{ik_y}]}{2} + S_1(-k) S_2(k) \frac{[1+e^{-ik_y}]}{2}$$

$$- S_1(k) S_1(-k) \left[\frac{e^{ik_x} + e^{-ik_x}}{2}\right]$$

$$+ S_2(k) S_2(-k) \left[\frac{e^{ik_x} + e^{-ik_x}}{2}\right] \Big]$$

$$= \sum_k [S_1(k) \quad S_2(k)] \begin{bmatrix} +\cos(k_x) & -\left[\frac{1+e^{ik_y}}{2}\right] \\ -\left[\frac{1+e^{-ik_y}}{2}\right] & -\cos(k_x) \end{bmatrix} \begin{bmatrix} S_1(-k) \\ S_2(-k) \end{bmatrix}$$

The eigenvalues of this matrix are :

$$E_\pm = \pm \sqrt{\cos^2(k_x) + \left|\frac{1+e^{ik_y}}{2}\right|^2}$$

$$= \pm \sqrt{\cos^2(k_x) + \frac{1+\cos(k_y)}{2}}$$

The low energy theory will be dicated by the eigenvalue lowest in energy $\Rightarrow$ modes at $(k_x = 0, k_y = 0)$ and $(k_x = \pi, k_y = 0)$ corresponding to E_- are the low energy modes.

The eigenvector corresponding to $(k_x = 0, k_y = 0)$ is found by

$$\begin{bmatrix} +1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -\sqrt{2} \begin{bmatrix} a \\ b \end{bmatrix}$$

where a, b are the amplitudes on 1, 2 sublattice. Solving, one finds, the unnormalized eigenvectors are $b \propto 2+\sqrt{2}$, $a \propto \sqrt{2}$.

Let's use the convention such that sublattice 1 corresponds to even rows and sublattice 2 to odd rows. Then this eigenvector can be written as:

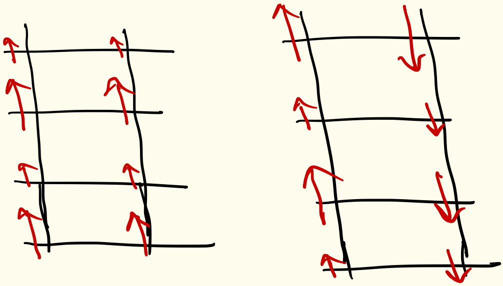
$$\psi^{(0,0)} = 1+\sqrt{2} - e^{i\pi y}$$

Similarly, one can solve for the eigenvector corresponding to $(k_x = \pi, k_y = 0)$

$$\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -\sqrt{2} \begin{bmatrix} a \\ b \end{bmatrix}.$$

$$\psi^{(\pi,0)} = \left[1+\sqrt{2} + e^{i\pi y}\right] e^{i\pi x}$$

These two modes look like



where the ratio of larger arrow to smaller arrow $= \sqrt{2}+1$.

Let's denote the fluctuations around these two modes as ϕ_0 and ϕ_π respectively, so that at low energies spin is given by,

$$S(x,y) = \phi_0(x,y)\,\psi^{(0,0)}(x,y) + \phi_\pi(x,y)\,\psi^{(\pi,0)}(x,y)$$

Let's study how ϕ_0 and ϕ_π transform under various lattice sym. This will determine the low-energy theory of ϕ_0, ϕ_π.

Note that: (i) ϕ_0, ϕ_π are slowly varying fields as compared to $\psi^{(0,0)}$ and $\psi^{(\pi,0)}$.

(ii) the guage choice breaks lattice syms. so one may need to perform a gauge transformation to restore it.

Translation along $\hat{x} \equiv T_x$.

The guage choice of J_{ij} does not break T_x, and therefore this operation does not require one to perform a guage transformation.

$$T_x S(x,y) T_x^{+} = S(x+1, y)$$

$$\Rightarrow T_x \phi_0 T_x^{+} \psi^{(0,0)}_{(x,y)} + T_x \phi_\pi T_x^{+} \psi^{(\pi,0)}_{(x,y)}$$

$$= \phi_0 (x,y) \psi^{(0,0)}(x+1,y)$$

$$+ \phi_\pi (x,y) \psi^{(\pi,0)}(x+1,y)$$

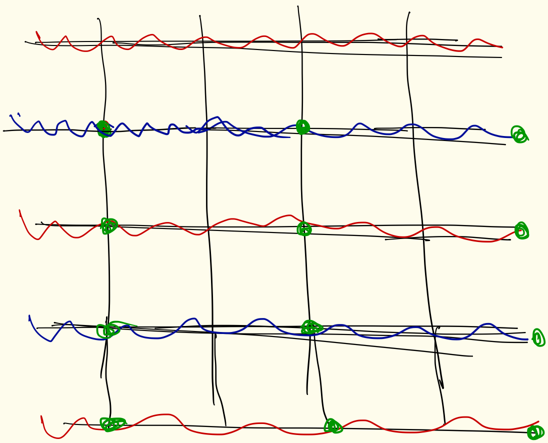
$$= \phi_0(x,y) \psi^{(0,0)}(x,y)$$

$$- \phi_\pi(x,y) \psi^{(\pi,0)}(x,y)$$

$$\Rightarrow \boxed{\begin{aligned} T_x \phi_0 T_x^{+} &= \phi_0 \\ T_y \phi_0 T_y^{+} &= -\phi_\pi \end{aligned}}$$

Translation along $\hat{y} \equiv T_y$.

The guage choice breaks T_y, therefore one needs to perform a guage transformation in addition to T_y to implement this sym.



Gauge transformation on green vertices maps the config. of J_{ij} before and after translation along $\hat{y}$ into each other.

$$T_y \, G \, S(x,y) \, G^{-1} \, T_y^{-1} = S(x, y+1).$$

where G is the above guage transformation.

Analytically, this guage transformation corresponds to $G \, S \, G^{-1} = e^{i\pi x} S$

$\Rightarrow$

$$T_y \, \phi_0 T_y^\dagger \left\{ [1+\sqrt{2}] - e^{i\pi y} \right\} e^{i\pi x}$$

$$+ \; T_y \, \phi_\pi \, T_y^\dagger \left\{ \underbrace{e^{2i\pi x}}_{=1} [(1+\sqrt{2}) + e^{i\pi y}] \right\}$$

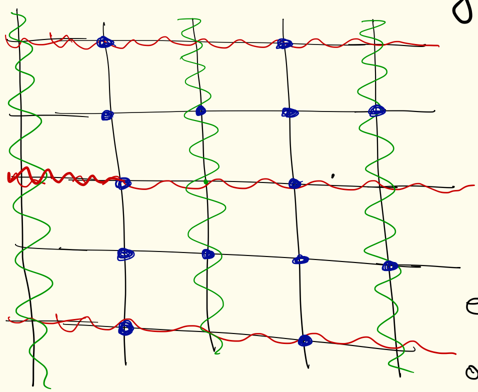
$$= \phi_0 \left[(1+\sqrt{2}) + e^{i\pi y} \right]$$

$$+ \; \phi_\pi \left\{ e^{i\pi x} [(1+\sqrt{2}) - e^{i\pi y}] \right\}$$

$$\Rightarrow \boxed{\begin{aligned} T_y \, \phi_0 T_y^\dagger &= \phi_\pi \\ T_y \, \phi_\pi T_y^\dagger &= \phi_0 \end{aligned}}$$

Rotation by $\pi/2 \equiv R_{\pi/2}$:

$R_{\pi/2}$ is also broken by the guage choice.



The guage transformation that maps the two config. into each other is:

even y : $s(x,y) \to e^{i\pi x} s(x,y)$

odd y : $s(x,y) \longrightarrow -s(x,y)$

Consider even y.

$$R_{\pi/2} \quad G \quad S(x,y) \quad G^{-1} \quad R_{\pi/2}^{-1}$$

$$= R_{\pi/2} \; S \; R_{\pi/2}^{-1} \quad e^{i\pi x}$$

$$= S(y, -x)$$

$$\Rightarrow R_{\pi/2} \, \phi_0 \, R_{\pi/2}^{-1} \; e^{i\pi x} \; \sqrt{2}$$

$$+ R_{\pi/2} \, \phi_\pi \, R_{\pi/2}^{-1} \, (2+\sqrt{2})$$

$$= \phi_0 \, [1 + \sqrt{2} - e^{i\pi x}]$$

$$+ \phi_\pi \, [1 + \sqrt{2} + e^{i\pi x}]$$

$$= e^{i\pi x} [\phi_\pi - \phi_0]$$

$$+ [1+\sqrt{2}] [\phi_0 + \phi_\pi]$$

$$\Rightarrow \boxed{\begin{aligned} R_{\pi/2} \, \phi_0 \, R_{\pi/2}^{-1} &= \frac{\phi_\pi - \phi_0}{\sqrt{2}} \\ R_{\pi/2} \, \phi_\pi \, R_{\pi/2}^{-1} &= \frac{\phi_\pi + \phi_0}{\sqrt{2}} \end{aligned}}$$

Ising symmetry $\equiv R_{Z_2}$

Under this $R_{Z_2} S(x,y) R_{Z_2} = -S(x,y)$. $\Rightarrow R_{Z_2}\phi_0 R_{Z_2}^{-1} = -\phi_0$

and $R_{Z_2}\phi_\pi R_{Z_2}^{-1} = -\phi_\pi$.

Defining $\varphi = \phi_0 + i\phi_\pi$, one can summarize our results:

$$T_x \varphi T_x^{-1} = \varphi^*$$

$$T_y \varphi T_y^{-1} = i\varphi^*$$

$$R_{\pi/2} \varphi R_{\pi/2}^{-1} = i\varphi^* e^{i\pi/4}$$

$$R_{Z_2} \varphi R_{Z_2}^{-1} = -\varphi$$

Thus, the low energy theory is:

$$\mathcal{L} = |\partial_\mu \varphi|^2 + r|\varphi|^2 + u|\varphi|^4 + v[\varphi^8 + \varphi^{*8}] + \dots$$

Writing $\varphi = m e^{i\theta}$,

$$\mathcal{L} = (\partial_\mu m)^2 + r\, m^2 + u\, m^4$$
$$+ \; v\, m^8 \cos(8\theta) + \ldots$$

The rotational sym. is broken by $\cos(8\theta)$ term, and in the sym broken phase (when J term dominates), one obtains a eight-fold ground state degeneracy.